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Fluid flows through fractured porous media along Beavers-Joseph interfaces

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Abstract

We study a fluid flow traversing a porous medium and obeying the Darcy's law in the case when this medium is fractured in blocks by an ε -periodic ($\varepsilon > 0$) distribution of fissures filled with a Stokes fluid. These two flows are coupled by a Beavers-Joseph type interface condition. The existence and uniqueness of this flow in our ε -periodic structure are proved. As the small period of the distribution shrinks to zero, we study the asymptotic behaviour of the flow when the permeability and the entire contribution on the interface of the Beavers-Joseph transfer coefficients are of unity order. We find the homogenized problem verified by the two-scale limits of the coupled velocities and pressures. It is well-posed and provides the corresponding classical homogenized problem.

Keywords: Fractured porous media, Stokes flow, Beavers-Joseph interface, Homogenization, Two-scale convergence.

2000 MSC: 35B27, 76M50, 76S05, 76T99

1. Introduction

One major achievement of homogenization theory was the mathematical justification of Darcy's law [23], considering the Stokes flow around an ε -periodic system of isolated solid obstacles. It was not until this non-connectedness assumption could be dropped out that the homogenization of phenomena in fractured media could be studied (see [17], [1] and [18]). The models of fluid flows through fractured porous media (see [5], [6], [22], [10] and [19]) are usually obtained by asymptotic methods, from the alteration of a homogeneous porous medium by a distribution of microscopic fissures.

Here, as the process at the microscopic scale takes place under the assumption of ε -periodicity, the study of its asymptotic behaviour (when $\varepsilon \rightarrow 0$) is amenable to the procedures of the homogenization theory. We are interested to which extent the Darcy's law, which is already a macroscopic approximation of a microscopic process, may be considered as existing besides the Stokes flow. The answer is brought in the form of an original model where we assume that the volumes of both media have the same order of magnitude.

Our study is devoted to the mathematical modelisation of the fractured material, namely, the slip boundary condition of Beavers-Joseph type is revisited in accordance with observed physical laws and the underlying arguments of the classical mathematical theory.

We consider an incompressible viscous fluid flow in a fractured porous media represented by a periodically structured domain consisting of two interwoven regions, separated by an interface. The first region represents the system of fissures which form the fracture, which is connected and where the flow is governed by the Stokes system. The second region, which is also connected, stands for the system of porous blocks, which have a certain permeability and where the flow is governed by Darcy's law. These two flows are coupled on the interface by the Saffman's variant [20] of the Beavers-Joseph condition [7], [13] which was confirmed by [12] as the limit of a homogenization process. Besides the continuity of the normal component of the velocity, it imposes the proportionality of the tangential velocity with the tangential component of the viscous stress on the fluid-side of the interface. We prove here the existence and uniqueness of this flow in our ε -periodic structure.

The system is rescaled in such a way that it becomes relevant that apart ε the asymptotic behaviour depends also on two parameters, depending on the permeability of the porous blocks and on the Beavers-Joseph transfer coefficient. As the small period of the distribution shrinks to zero, we study the asymptotic behaviour of the flow when the permeability of the porous blocks is of unity order and the Beavers-Joseph transfer coefficient is of ε -order, balancing the measure of the interface, which is of ε^{-1} -order. Our main result is the identification of the homogenized problem. It is a well-posed problem, verified by the two-scale limits of the velocity, Darcy's pressure and a second Stokes velocity responding for the microscopic exchange of mass through the interface. Finally, by eliminating the second Stokes velocity, we find the corresponding classical homogenized problem.

The paper is organized as follows.

In Section 2, we present the ε -periodic structure, set the corresponding

problem and prove the existence and uniqueness of its weak solution.

Section 3 is devoted to the a priori estimates, which serve as departure point for adapting the compactness results of the two-scale convergence theory (see [3] and [14]).

In Section 4 we find the so-called two-scale homogenized problem. We prove that it is a well-posed problem. As a corollary, we get the corresponding classical homogenized problem.

2. The flow through the ε -periodic structure

Let Ω be an open connected bounded set in \mathbb{R}^N ($N \geq 2$), locally located on one side of the boundary $\partial\Omega$, a Lipschitz manifold composed of a finite number of connected components.

Let Y_f be a Lipschitz open connected subset of the unit cube $Y =]0, 1[^N$, such that the intersections of ∂Y_f with ∂Y are reproduced identically on the opposite faces of the cube and $0 \notin \bar{Y}_f$. The outward normal on ∂Y_f is denoted by ν . Repeating Y by periodicity, we assume that the reunion of all the \bar{Y}_f parts, denoted by \mathbb{R}_f^N , is a connected domain in \mathbb{R}^N with a boundary of class \mathcal{C}^2 . Defining $Y_s = Y \setminus \bar{Y}_f$, we assume also that the reunion of all the \bar{Y}_s parts is a connected domain in \mathbb{R}^N .

For any $\varepsilon \in]0, 1[$ we denote

$$Z_\varepsilon = \{k \in \mathbb{Z}^N, \varepsilon k + \varepsilon Y \subseteq \Omega\} \quad (2.1)$$

$$I_\varepsilon = \{k \in Z_\varepsilon, \varepsilon k \pm \varepsilon e_i + \varepsilon Y \subseteq \Omega, \forall i \in \overline{1, N}\} \quad (2.2)$$

where e_i are the unit vectors of the canonical basis in \mathbb{R}^N .

Finally, we define the system of fissures by

$$\Omega_{\varepsilon f} = \text{int} \left(\bigcup_{k \in I_\varepsilon} (\varepsilon k + \varepsilon \bar{Y}_f) \right) \quad (2.3)$$

and the porous matrix of our structure by $\Omega_{\varepsilon s} = \Omega \setminus \bar{\Omega}_{\varepsilon f}$. The interface between the porous blocks and the fluid is denoted by $\Gamma_\varepsilon = \partial\Omega_{\varepsilon f}$. Its normal is:

$$\nu^\varepsilon(x) = \nu \left(\frac{x}{\varepsilon} \right), \quad x \in \Gamma_\varepsilon \quad (2.4)$$

where ν has been periodically extended to \mathbb{R}^N .

Let us remark that $\Omega_{\varepsilon s}$ and $\Omega_{\varepsilon f}$ are connected and that the fracture ratio of this structure is given by

$$m = |Y_f| \in]0, 1[, \quad \text{as} \quad \frac{|\Omega_{\varepsilon f}|}{|\Omega|} \rightarrow m \quad \text{when} \quad \varepsilon \rightarrow 0. \quad (2.5)$$

To the previous structure we associate a model of fluid flow through a fractured porous medium by assuming that there is a filtration flow in $\Omega_{\varepsilon s}$ obeying the Darcy's law and that there is a viscous flow in $\Omega_{\varepsilon f}$ governed by the Stokes system. These two flows are coupled by a Saffman's variant [20] of the Beavers-Joseph condition [7], [13]. This system is completed by an impermeability condition on $\partial\Omega$:

$$\operatorname{div} v^{\varepsilon s} = 0 \quad \text{in } \Omega_{\varepsilon s} \quad (2.6)$$

$$\mu_\varepsilon v^{\varepsilon s} = K^\varepsilon (g^\varepsilon - \nabla p^{\varepsilon s}) \quad \text{in } \Omega_{\varepsilon s}, \quad (2.7)$$

$$\operatorname{div} v^{\varepsilon f} = 0 \quad \text{in } \Omega_{\varepsilon f}, \quad (2.8)$$

$$\sigma_{ij}^\varepsilon = -p^{\varepsilon f} \delta_{ij} + 2\mu_\varepsilon e_{ij}(v^{\varepsilon f}) \quad \text{in } \Omega_{\varepsilon f} \quad (2.9)$$

$$-\frac{\partial}{\partial x_j} \sigma_{ij}^\varepsilon = g_i^\varepsilon \quad \text{in } \Omega_{\varepsilon f} \quad (2.10)$$

$$v^{\varepsilon s} \cdot \nu^\varepsilon = v^{\varepsilon f} \cdot \nu^\varepsilon \quad \text{on } \Gamma_\varepsilon, \quad (2.11)$$

$$-p^{\varepsilon s} \nu_i^\varepsilon - \sigma_{ij}^\varepsilon \nu_j^\varepsilon = \alpha_\varepsilon \mu_\varepsilon (Tr K^\varepsilon)^{-1/2} (v_i^{\varepsilon f} - (v^{\varepsilon f} \cdot \nu^\varepsilon) \nu_i^\varepsilon) \quad \text{on } \Gamma_\varepsilon, \quad (2.12)$$

$$v^{\varepsilon s} \cdot n = 0 \quad \text{on } \partial\Omega, \quad n \text{ the outward normal on } \partial\Omega, \quad (2.13)$$

where $v^{\varepsilon s}$, $v^{\varepsilon f}$ and $p^{\varepsilon s}$, $p^{\varepsilon f}$ stand for the corresponding velocities and pressures, $\mu^\varepsilon > 0$ is the viscosity of the fluid, $K^\varepsilon \in L^\infty(\Omega)^{N \times N}$ is the positively-defined tensor of permeability, $\alpha_\varepsilon \in L^\infty(\Omega)$ is the positive non-dimensional Beavers-Joseph number, $g^\varepsilon \in L^2(\Omega)^N$ is the exterior force and $e(v)$ denotes the symmetric tensor of the velocity gradient defined by

$$e_{ij}(v) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

As usual, we use the notations:

$$H_0(\operatorname{div}, \Omega) = \{v \in H(\operatorname{div}, \Omega), \quad v \cdot \nu = 0 \quad \text{on } \partial\Omega\} \quad (2.14)$$

$$L_0^2(\Omega) = \{p \in L^2(\Omega), \quad \int_\Omega p = 0\} \quad (2.15)$$

$$V_0(\operatorname{div}, \Omega) = \{v \in H_0(\operatorname{div}, \Omega), \quad \operatorname{div} v = 0 \quad \text{in } \Omega\} \quad (2.16)$$

Next, we define

$$H_\varepsilon = \{v \in H_0(\operatorname{div}, \Omega), \quad v \in H^1(\Omega_{\varepsilon f})^N\}, \quad (2.17)$$

the Hilbert space endowed with the scalar product

$$(u, v)_{H_\varepsilon} = \int_{\Omega_{\varepsilon s}} uv + \int_{\Omega_{\varepsilon s}} \operatorname{div} u \operatorname{div} v + \int_{\Omega_{\varepsilon f}} e(u)e(v) + \varepsilon \int_{\Gamma_\varepsilon} (\gamma^\varepsilon u - (\gamma_\nu^\varepsilon u) \nu^\varepsilon) \gamma^\varepsilon v \quad (2.18)$$

where γ^ε and γ_ν^ε denote respectively the trace and the normal trace operators on Γ_ε with respect to $\Omega_{\varepsilon f}$. Its corresponding subspace of incompressible velocities is

$$V_\varepsilon = \{v \in V_0(\operatorname{div}, \Omega), \quad v \in H^1(\Omega_{\varepsilon f})^N\} \quad (2.19)$$

A useful property of the present structure is the existence of a bounded extension operator similar to that introduced in [8], [9] and [2] in the case of isolated fractures.

Theorem 2.1. *There exists an extension operator $P_\varepsilon : H^1(\Omega_{\varepsilon f}) \rightarrow H^1(\Omega)$ such that*

$$P_\varepsilon u = u \quad \text{in} \quad \Omega_{\varepsilon f} \quad (2.20)$$

$$|e(P_\varepsilon u)|_{L^2(\Omega)} \leq C |e(u)|_{L^2(\Omega_{\varepsilon f})}, \quad \forall u \in H^1(\Omega_{\varepsilon f}) \quad (2.21)$$

where C is independent of ε .

A straightforward consequence, via the corresponding Korn inequality, is

Lemma 2.2. *There exists some constant $C > 0$, independent of ε , such that*

$$|u|_{H^1(\Omega_{\varepsilon f})} \leq C |u|_{H_\varepsilon}, \quad \forall u \in H_\varepsilon. \quad (2.22)$$

Denoting

$$A^\varepsilon = (\operatorname{Tr} K^\varepsilon)(K^\varepsilon)^{-1}, \quad (2.23)$$

$$\beta_\varepsilon = (\operatorname{Tr} K^\varepsilon)^{1/2} > 0 \quad (2.24)$$

and using the positivity of K^ε , we can assume without loss of generality that

$$\exists a_0 > 0 \quad \text{such that} \quad A_{ij}^\varepsilon(\cdot) \xi_i \xi_j \geq a_0 |\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad \text{a.e. in } \Omega. \quad (2.25)$$

Rescaling the velocity by

$$u^\varepsilon = \begin{cases} u^{\varepsilon s} & \text{in } \Omega_{\varepsilon s} \\ u^{\varepsilon f} & \text{in } \Omega_{\varepsilon f} \end{cases} = \frac{\mu_\varepsilon}{(\operatorname{Tr} K^\varepsilon)} \begin{cases} v^{\varepsilon s} & \text{in } \Omega_{\varepsilon s} \\ v^{\varepsilon f} & \text{in } \Omega_{\varepsilon f} \end{cases} \quad (2.26)$$

then, for any $u, v \in H_\varepsilon$ and $q \in L_0^2(\Omega)$, we define

$$a_\varepsilon(u, v) = \int_{\Omega_{\varepsilon s}} A^\varepsilon uv + \beta_\varepsilon^2 \int_{\Omega_{\varepsilon f}} e(u)e(v) + \beta_\varepsilon \int_{\Gamma_\varepsilon} \alpha_\varepsilon(\gamma^\varepsilon u - (\gamma_\nu^\varepsilon u)\nu^\varepsilon)\gamma^\varepsilon v \quad (2.27)$$

$$b_\varepsilon(q, v) = - \int_{\Omega} q \operatorname{div} v. \quad (2.28)$$

We see that if the pair $(u^\varepsilon, p^\varepsilon)$ is a smooth solution of the problem (2.6)–(2.13), then it is also a solution of the following problem: To find $(u^\varepsilon, p^\varepsilon) \in H_\varepsilon \times L_0^2(\Omega)$ such that

$$a_\varepsilon(u^\varepsilon, v) + b_\varepsilon(p^\varepsilon, v) = \int_{\Omega} g^\varepsilon v, \quad \forall v \in H_\varepsilon \quad (2.29)$$

$$b_\varepsilon(q, u^\varepsilon) = 0, \quad \forall q \in L_0^2(\Omega) \quad (2.30)$$

Theorem 2.3. *There exists a unique pair $(u^\varepsilon, p^\varepsilon) \in H_\varepsilon \times L_0^2(\Omega)$ solution of (2.29)–(2.30).*

Proof. As $H_0^1(\Omega)$ is obviously included in H_ε , the following inf-sup condition is easily satisfied by b_ε :

$$\exists C_1^\varepsilon > 0 \quad \text{such that} \quad \inf_{q \in L_0^2(\Omega)} \sup_{v \in H_\varepsilon} \frac{b_\varepsilon(q, v)}{|v|_{H_\varepsilon} |q|_{L_0^2(\Omega)}} \geq C_1^\varepsilon.$$

The positivity conditions (2.24) and (2.25) imply that

$$\exists C_2^\varepsilon > 0 \quad \text{such that} \quad a_\varepsilon(v, v) \geq C_2^\varepsilon |v|_{H_\varepsilon}^2, \quad \forall v \in H_\varepsilon,$$

that is the V_ε -ellipticity of a_ε . As we also have

$$V_\varepsilon = \{v \in H_\varepsilon, \quad b_\varepsilon(q, v) = 0, \quad \forall q \in L_0^2(\Omega)\}, \quad (2.31)$$

the proof is completed by Corollary 4.1, Ch. 1 of [11]. \blacksquare

In the rest of the paper we shall study the asymptotic behaviour (when $\varepsilon \rightarrow 0$) of $(u^\varepsilon, p^\varepsilon)$, the unique solution of (2.29)–(2.30).

As A_ε defined by (2.23) is of ε^0 -order, we assume that

$$\exists A \in L_{\text{per}}^\infty(Y)^{N^2} \quad \text{such that} \quad A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right), \quad x \in \Omega \quad (2.32)$$

$$\exists g \in L^2(\Omega)^N \quad \text{such that} \quad g^\varepsilon \rightarrow g \quad \text{strongly in} \quad L^2(\Omega). \quad (2.33)$$

Because $|\Gamma_\varepsilon|$ is of ε^{-1} -order, we expect that macroscopic effects of the Beavers-Joseph condition will appear only when $\alpha_\varepsilon\beta_\varepsilon$ is of ε^1 -order (see [15]). Therefore, we shall work under the hypothesis:

$$\exists \alpha \in C_{\text{per}}^1(Y) \text{ and } \alpha_0 > 0 \text{ such that } \varepsilon^{-1}\beta_\varepsilon\alpha_\varepsilon(x) = \alpha\left(\frac{x}{\varepsilon}\right) \geq \alpha_0, \quad x \in \Omega. \quad (2.34)$$

Thus, for the study of the asymptotic behaviour it remains only the order of β_ε to be taken into account.

In the sequel we shall study the case when β_ε is of unity order. Without loss of generality we can consider from now on that

$$\exists \beta > 0 \quad \text{such that} \quad \beta_\varepsilon = \beta, \quad \forall \varepsilon > 0. \quad (2.35)$$

3. A priori estimates and two-scale convergences

From now on, for any function φ defined on $\Omega \times Y$ we shall use the notations

$$\varphi^h = \varphi|_{\Omega \times Y_h}, \quad \tilde{\varphi}^h = \frac{1}{|Y_h|} \int_{Y_h} \varphi(\cdot, y) dy, \quad h \in \{s, f\}, \quad (3.1)$$

$$\tilde{\varphi} = \int_Y \varphi(\cdot, y) dy, \quad \text{that is} \quad \tilde{\varphi} = (1 - m)\tilde{\varphi}^s + m\tilde{\varphi}^f. \quad (3.2)$$

Also, for any sequence $(\varphi^\varepsilon)_\varepsilon$, bounded in $L^2(\Omega \times Y)$, we denote

$$\varphi^\varepsilon \xrightarrow{2} \varphi$$

iff φ^ε is two-scale convergent to $\varphi \in L^2(\Omega \times Y)$ in the sense of [3].

As usual, the asymptotic study starts by the search of a priori estimations.

Noticing that $u^\varepsilon \in V_\varepsilon$ and setting $v = u^\varepsilon$ in (2.29) we get

$$|u^\varepsilon|_{H_\varepsilon}^2 \leq C|u^\varepsilon|_{L^2(\Omega)} \quad (3.3)$$

Applying (2.22) we find that

$$\{u^\varepsilon\}_\varepsilon \text{ is bounded in } V_\varepsilon \text{ and in } V_0(\text{div}, \Omega). \quad (3.4)$$

$$|u^\varepsilon|_{H^1(\Omega_{\varepsilon f})} \leq C, \quad C \text{ being independent of } \varepsilon \quad (3.5)$$

It follows that $\exists u \in L^2(\Omega \times Y)^N$ such that, on some subsequence

$$u^\varepsilon \xrightarrow{2} u \quad (3.6)$$

$$u^\varepsilon \rightharpoonup \int_Y u(\cdot, y) dy \in V_0(\operatorname{div}, \Omega) \quad \text{weakly in} \quad L^2(\Omega)^N \quad (3.7)$$

Denoting $\chi_{\varepsilon f}(x) = \chi_f\left(\frac{x}{\varepsilon}\right)$ and $\chi_{\varepsilon s}(x) = \chi_s\left(\frac{x}{\varepsilon}\right)$, where χ_f and χ_s are the characteristic functions of Y_f and Y_s in Y , we see that $(\chi_{\varepsilon s} u^\varepsilon)_\varepsilon$, $(\chi_{\varepsilon f} u^\varepsilon)_\varepsilon$ and $\left(\chi_{\varepsilon f} \frac{\partial u^\varepsilon}{\partial x_i}\right)_\varepsilon$ are bounded in $(L^2(\Omega))^N$, $\forall i \in \{1, 2, \dots, N\}$.

Using the compactness result of [16], it follows that $\exists \eta_i \in L^2(\Omega \times Y)^N$ such that, on some subsequence of (3.6)–(3.7) convergences, we have also

$$u^\varepsilon \xrightarrow{2} u \quad (3.8)$$

$$\chi_{\varepsilon f} \nabla u_i^\varepsilon \xrightarrow{2} \eta_i \quad (3.9)$$

Denoting by

$$\tilde{H}_{\text{per}}^1(Y_f) = \{\varphi \in H_{\text{loc}}^1(\mathbb{R}_f^N), \quad \varphi \text{ is } Y\text{-periodic}, \quad \int_{Y_f} \varphi = 0\} \quad (3.10)$$

we can present a first result.

Lemma 3.1. $u^f = \tilde{u}^f \in H_0^1(\Omega)^N$ and there exists $w \in L^2(\Omega, (\tilde{H}_{\text{per}}^1(Y_f))^N)$ such that

$$\eta_i = \chi_f \left(\nabla u_i^f + \nabla_y w_i \right) \quad (3.11)$$

Proof. Let $\psi \in \mathcal{D}(\Omega, C_{\text{per}}^\infty(Y_f))$ with $\int_Y \psi = 0$ in Ω . Let us consider $\varphi \in \mathcal{D}(\Omega, H_{\text{per}}^1(Y_f)^N)$ satisfying

$$\operatorname{div}_y \varphi = \psi \quad \text{in} \quad \Omega \times Y_f \quad (3.12)$$

$$\varphi \cdot \nu = 0 \quad \text{in} \quad \Omega \times \Gamma. \quad (3.13)$$

Defining $\varphi^\varepsilon(x) = \varphi\left(x, \frac{x}{\varepsilon}\right)$ we find that $\varphi^\varepsilon \in H^1(\Omega_{\varepsilon f})^N$ and $\gamma_\nu^\varepsilon \varphi^\varepsilon = 0$ on Γ_ε . As $u^\varepsilon \in H^1(\Omega_{\varepsilon f})^N$ with $u^\varepsilon = 0$ on $\partial\Omega \cap \partial\Omega_{\varepsilon f}$ it follows

$$\begin{aligned} & \int_{\Omega_{\varepsilon f}} u^\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \\ & = -\varepsilon \int_{\Omega_{\varepsilon f}} \frac{\partial u^\varepsilon}{\partial x_i}(x) \varphi_i\left(x, \frac{x}{\varepsilon}\right) dx - \varepsilon \int_{\Omega_{\varepsilon f}} u^\varepsilon(x) (\operatorname{div}_x \varphi)\left(x, \frac{x}{\varepsilon}\right) dx. \end{aligned} \quad (3.14)$$

Passing at the limit on the subsequence on which (3.6)–(3.9) hold, we find

$$\int_{\Omega \times Y_f} u(x, y) \psi(x, y) dx dy = 0. \quad (3.15)$$

It follows that $\exists v \in L^2(\Omega)^N$ such that $u^f = v$ and hence

$$\chi_{\varepsilon f} u^\varepsilon \rightharpoonup m u^f \quad \text{weakly in } L^2(\Omega)^N. \quad (3.16)$$

But from Lemma A.3 [4] we know that $\exists \hat{v} \in H_0^1(\Omega)$ such that, by extracting a subsequence of (3.16) we have

$$\chi_{\varepsilon f} u^\varepsilon \rightharpoonup m \hat{v} \quad \text{weakly in } L^2(\Omega)^N, \quad (3.17)$$

that is $u^f = \hat{v} \in H_0^1(\Omega)$.

It remains to prove (3.11). First, we remark that $\eta_i|_{\Omega \times Y_s} = 0$.

Let us introduce

$$\begin{aligned} V_0^{\text{per}}(\text{div}, Y_f) &= \{\varphi \in H_{\text{loc}}(\text{div}, \mathbb{R}_f^N), \text{div}_y \varphi = 0 \text{ in } R_f^N, \\ &\quad \varphi \cdot \nu = 0 \text{ on } \Gamma, \varphi \text{ is } Y\text{-periodic}\}. \end{aligned} \quad (3.18)$$

In the classical way (see Th.2.7, Ch.I [11]), we find that the orthogonal space of $V_0^{\text{per}}(\text{div}, Y_f)$ in $L^2(\Omega \times Y)$ is:

$$\nabla \tilde{H}_{\text{per}}^1(Y_f) = \{\nabla q, \quad q \in \tilde{H}_{\text{per}}^1(Y_f)\} \quad (3.19)$$

Let $\psi \in L^2(\Omega, V_0^{\text{per}}(\text{div}, Y_f))$; denoting $\psi^\varepsilon(x) = \psi\left(x, \frac{x}{\varepsilon}\right)$, we have

$$\psi^\varepsilon \in H^1(\Omega_{\varepsilon f})^N \quad \text{and} \quad \psi^\varepsilon \nu^\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon \quad (3.20)$$

$$\int_{\Omega_{\varepsilon f}} \nabla u_i^\varepsilon(x) \psi^\varepsilon(x) dx = - \int_{\Omega_{\varepsilon f}} u_i^\varepsilon(x) (\text{div}_x \psi) \left(x, \frac{x}{\varepsilon}\right) dx. \quad (3.21)$$

Using the two-scale convergences (3.6)–(3.9) we find

$$\begin{aligned} \int_{\Omega \times Y_f} \eta_i(x, y) \psi(x, y) dx dy &= - \int_{\Omega} u^f(x) \text{div}_x \left(\int_{Y_f} \psi(x, y) dy \right) dx = \\ &= \int_{\Omega \times Y_f} \nabla u_i^f(x) \psi(x, y) dx dy \end{aligned} \quad (3.22)$$

which yields $(\eta_i - \nabla u_i^f) \in L^2(\Omega, \nabla \tilde{H}_{\text{per}}^1(Y_f))$ for any $i \in \{1, \dots, N\}$ and the proof is completed. \blacksquare

In the following, we sum the convergence results obtained until now.

Theorem 3.2. *There exist $w \in L^2(\Omega, \tilde{H}_{\text{per}}^1(Y_f)^N)$ and $u \in L^2(\Omega, V_0^{\text{per}}(\text{div}, Y_f))$ with $u^f \in H_0^1(\Omega)$ such that the following convergences hold on some subsequence:*

$$u^\varepsilon \xrightarrow{2} u \quad (3.23)$$

$$\chi_{\varepsilon f} \nabla u_i^\varepsilon \xrightarrow{2} \chi_f (\nabla u_i^f + \nabla_y w_i), \quad \forall i \quad (3.24)$$

Moreover, we have

$$\gamma_n \tilde{u}^s = 0 \quad \text{on} \quad \partial\Omega \quad (3.25)$$

$$(1 - m) \text{div} \tilde{u}^s + m \text{div} u^f = 0 \quad \text{in} \quad \Omega \quad (3.26)$$

$$\text{div}_y u = 0 \quad \text{in} \quad \Omega \times Y \quad (3.27)$$

$$\text{div}_y w + \text{div} u^f = 0 \quad \text{in} \quad \Omega \times Y_f \quad (3.28)$$

where γ_n denotes the normal trace on $\partial\Omega$.

Proof. The convergences (3.23)–(3.24) are straight consequences of (3.8)–(3.9) and Lemma 3.1. The properties (3.25)–(3.26) follow from (3.7). Now let $\varphi \in \mathcal{D}(\Omega, H_{\text{per}}^1(Y))$ with

$$\varphi = 0 \quad \text{in} \quad \Omega \times Y_f. \quad (3.29)$$

Defining $\varphi^\varepsilon(x) = \varphi\left(x, \frac{x}{\varepsilon}\right)$, we notice that $\varphi^\varepsilon \in \mathcal{D}(\Omega)$ and hence

$$\begin{aligned} 0 &= \varepsilon \int_{\Omega} u^\varepsilon \nabla \varphi^\varepsilon(x) dx = \\ &= \int_{\Omega} u^\varepsilon(x) (\nabla_y \varphi)\left(x, \frac{x}{\varepsilon}\right) dx + \varepsilon \int_{\Omega} u^\varepsilon(x) (\nabla_x \varphi)\left(x, \frac{x}{\varepsilon}\right) dx. \end{aligned} \quad (3.30)$$

Using (3.23), we pass to the limit and get

$$\int_{\Omega \times Y_s} u(x, y) \nabla_y \varphi(x, y) dx dy = 0. \quad (3.31)$$

Hence, we have since now:

$$\text{div}_y u^s = 0 \quad \text{in} \quad \Omega \times Y_s \quad \text{and} \quad \text{div}_y u^f = 0 \quad \text{in} \quad \Omega \times Y_f.$$

Let $\varphi \in \mathcal{D}(\Omega, H_{\text{per}}^1(Y))$. Denoting

$$\varphi^\varepsilon(x) = \varepsilon \varphi\left(x, \frac{x}{\varepsilon}\right) \quad \text{for a.e. } x \in \Omega, \quad (3.32)$$

we get $\varphi^\varepsilon \in H_0^1(\Omega)$. As $u^\varepsilon \in V_\varepsilon$, we easily obtain

$$\int_{\Omega} (\chi_{\varepsilon s} u^\varepsilon)(x) (\nabla_y \varphi) \left(x, \frac{x}{\varepsilon}\right) dx + \int_{\Omega} (\chi_{\varepsilon f} u^\varepsilon)(x) (\nabla_y \varphi) \left(x, \frac{x}{\varepsilon}\right) dx = O(\varepsilon) \quad (3.33)$$

Using (3.23) we find that

$$\int_{\Omega \times Y_s} u(x, y) (\nabla_y \varphi)(x, y) dx dy + \int_{\Omega \times Y_f} v(x) (\nabla_y \varphi)(x, y) dx dy = 0 \quad (3.34)$$

which implies

$$\int_{\Omega \times \Gamma} (\gamma_\nu v - \gamma_\nu u) \varphi = 0 \quad (3.35)$$

where γ_ν denotes the normal trace on Γ ; the property (3.27) is proved.

The property (3.28) can be proved similarly. \blacksquare

Theorem 3.3. *There exists $p \in L_0^2(\Omega \times Y)$ with $p^s = \tilde{p}^s \in H^1(\Omega)$, such that on some subsequence we have*

$$p^\varepsilon \xrightarrow{2} p. \quad (3.36)$$

Proof. As $p^\varepsilon \in L_0^2(\Omega)$, it follows that there exists $\varphi^\varepsilon \in H_0^1(\Omega)^N$ such that

$$\operatorname{div} \varphi^\varepsilon = p^\varepsilon \quad \text{in } \Omega \quad (3.37)$$

$$|\nabla \varphi^\varepsilon|_{L^2(\Omega)} \leq C |p^\varepsilon|_{L^2(\Omega)} \quad \text{with } C \text{ independent of } \varepsilon. \quad (3.38)$$

For any $\varphi \in H_0^1(\Omega)^N$, we have like in [10]:

$$\varepsilon^{1/2} |\varphi|_{L^2(\Gamma_\varepsilon)} \leq C \left(\varepsilon |\nabla \varphi|_{L^2(\Omega_{\varepsilon f})} + |\varphi|_{L^2(\Omega_{\varepsilon f})} \right). \quad (3.39)$$

Combining it with (2.22) we get, via (3.39):

$$\varepsilon^{1/2} |\gamma^\varepsilon \varphi^\varepsilon - (\gamma_\nu^\varepsilon \varphi^\varepsilon) \nu^\varepsilon|_{L^2(\Gamma_\varepsilon)} \leq C |\nabla \varphi^\varepsilon|_{L^2(\Omega_{\varepsilon f})} \leq C |p^\varepsilon|_{L^2(\Omega)}. \quad (3.40)$$

Then, putting $v = \varphi^\varepsilon$ in (2.29) we obtain immediately

$$|p^\varepsilon|_{L^2(\Omega)} \leq C, \quad \text{for some } C > 0 \text{ independent of } \varepsilon. \quad (3.41)$$

Using the compacity result of [16], we find that there exists $p \in L_0^2(\Omega \times Y)$ such that the convergence (3.36) holds on some subsequence.

Now, choosing $v \in H_\varepsilon$ in (2.29) with $v = 0$ in $\Omega_{\varepsilon f}$, we get

$$-\nabla p^{\varepsilon s} = A^\varepsilon u^\varepsilon - g^\varepsilon \quad \text{in } \Omega_{\varepsilon s}, \quad (3.42)$$

that is, $(\chi_{\varepsilon s} p^{\varepsilon s})_\varepsilon$ is bounded in $H^1(\Omega)$. Then, by standard procedures we prove that $p^s = \tilde{p}^s \in H^1(\Omega)$ and the proof is completed. \blacksquare

4. The two-scale homogenized problem

Now, we introduce the so-called two-scale homogenized problem, verified by the limits u , w and p , given by Theorems 3.2–3.3. By proving that this problem is well-posed, it turns that all the convergences in Theorems 3.2–3.3 and so forth hold on the entire sequence. We conclude that the asymptotic behaviour of $((u^\varepsilon, w^\varepsilon), p^\varepsilon)$ is completely described by $((u, w), p)$, the unique solution of the homogenized problem.

Denoting

$$\begin{aligned} H = \{u \in L^2(\Omega \times Y), \ u^f = \tilde{u}^f \in H_0^1(\Omega)^N, \ \tilde{u} \in H_0(\text{div}, \Omega), \\ \text{div}_y u = 0 \text{ in } \Omega \times Y\} \end{aligned} \quad (4.1)$$

$$V = \{u \in H, \ \text{div} \tilde{u} = 0\}$$

we introduce the Hilbert space

$$X = H \times L^2(\Omega, \tilde{H}_{\text{per}}^1(Y_f)^N) \quad (4.2)$$

endowed with the scalar product:

$$((u, w), (v, z))_X = \int_{\Omega \times Y_s} u v + \int_{\Omega} \text{div} \tilde{u} \text{div} \tilde{v} + \int_{\Omega \times Y_f} (e(u) + e_y(w))(e(v) + e_y(z)).$$

The last associated spaces are:

$$M = \{q \in L_0^2(\Omega, L_{\text{per}}^2(Y)), \ q^s = \tilde{q}^s \in H^1(\Omega)\}$$

$$X_0 = \{(u, w) \in X, \ \text{div} \tilde{u} = 0 \text{ in } \Omega, \ \text{div}_y w + \text{div} u^f = 0 \text{ in } \Omega \times Y_f\}$$

The two-scale homogenized problem is the following:

To find $(v, z) \in X$ and $q \in M$ such that

$$a((v, z), (\varphi, \psi)) + b(q, (\varphi, \psi)) = \int_{\Omega} g \tilde{\varphi}, \quad \forall (\varphi, \psi) \in X \quad (4.3)$$

$$b(\pi, (v, z)) = 0, \quad \forall \pi \in M \quad (4.4)$$

where a and b are defined by

$$a((v, z), (\varphi, \psi)) = \int_{\Omega \times Y_s} A v \varphi + \beta^2 \int_{\Omega \times Y_f} (e(v) + e_y(z))(e(\varphi) + e_y(\psi)) +$$

$$\begin{aligned}
& +\beta \int_{\Omega \times \Gamma} \alpha(\gamma v - (\gamma_\nu v)\nu) \varphi \\
b(\pi, (v, z)) &= \int_{\Omega \times Y} \pi \operatorname{div}_x v - \int_{\Omega \times \Gamma} \pi^s(z \cdot \nu) + \int_{\Omega \times Y_f} \pi \operatorname{div}_y z
\end{aligned}$$

Theorem 4.1. *u, w and p introduced by the Theorems 3.2–3.3 form a pair $((u, w), p) \in X \times M$ which verifies the homogenized problem (4.3)–(4.4).*

Proof. From Theorem 3.2 we see that (4.3) is readily verified.

Let $\varphi \in \mathcal{D}(\Omega, C_{\text{per}}^\infty(Y))^N$, $\psi \in \mathcal{D}(\Omega, C_{\text{per}}^\infty(Y_f))^N$ such that $(\varphi, \psi) \in X$. Let $\hat{\psi}$ a prolongation of ψ to $\mathcal{D}(\Omega, \tilde{H}_{\text{per}}(\operatorname{div}, Y))$, which can be done, for instance, by considering a certain Neumann problem in Y_s . Denoting, as usual, $\varphi^\varepsilon(x) = \varphi\left(x, \frac{x}{\varepsilon}\right)$ and $\psi^\varepsilon(x) = \hat{\psi}\left(x, \frac{x}{\varepsilon}\right)$, we can set $v(x) = \varphi^\varepsilon(x) + \varepsilon\psi^\varepsilon(x)$ in (2.29). Passing to the limit with $\varepsilon \rightarrow 0$ and using the two-scale convergences of Theorems 3.2–3.3, we obtain:

$$\begin{aligned}
b_\varepsilon(p^\varepsilon, \varphi^\varepsilon + \varepsilon\psi^\varepsilon) &\rightarrow - \int_{\Omega \times Y_s} p(\operatorname{div}_x \varphi + \operatorname{div}_y \hat{\psi}) - \int_{\Omega \times Y_f} p(\operatorname{div}_x \varphi + \operatorname{div}_y \psi) = \\
&= b(p, (\varphi, \psi)),
\end{aligned}$$

where we have used also $p^s = \tilde{p}^s$ and $\hat{\psi}^s \cdot \nu = \psi^f \cdot \nu$ on Γ .

Next, we obtain:

$$a_\varepsilon(u^\varepsilon, \varphi^\varepsilon + \varepsilon\psi^\varepsilon) \rightarrow a((u, w), (\varphi, \psi)),$$

all the convergences being straightforward, except the one involving Γ_ε ; we present it here.

Let $\varphi \in \mathcal{D}(\Omega)^N$; for any $i, j \in \{1, 2, \dots, N\}$, there exists $G_{ij} \in \mathcal{D}(\Omega, C_{\text{per}}^1(Y_f))$ such that

$$G_{ij}(x, y) = (\varphi_i(x) - \varphi_k(x)\nu_k(y)\nu_i(y))\nu_j(y) \quad \text{on } \Gamma. \quad (4.5)$$

This follows from the C^1 -property on Γ of the right-hand side of (4.5) and from the smoothness of its prolongation with zero on ∂Y_f . Thus, $G_{ij}^\varepsilon(\cdot) = G_{ij}\left(\cdot, \frac{\cdot}{\varepsilon}\right) \in C_0^1(\Omega_{\varepsilon f})$ and we have

$$\begin{aligned}
\varepsilon \int_{\Gamma_\varepsilon} \alpha_\varepsilon(\gamma^\varepsilon u^\varepsilon - (\gamma_\nu^\varepsilon u^\varepsilon)\nu^\varepsilon) \gamma^\varepsilon \varphi &= \varepsilon \int_{\partial \Omega_{\varepsilon f}} \alpha_\varepsilon G_{ij}^\varepsilon \gamma^\varepsilon u_i^\varepsilon \nu_j^\varepsilon d\sigma = \\
&= \int_{\Omega_{\varepsilon f}} \frac{\partial(\alpha G_{ij})}{\partial y_j} \left(x, \frac{x}{\varepsilon}\right) u_i^\varepsilon(x) dx + O(\varepsilon)
\end{aligned}$$

$$\rightarrow \int_{\Omega} v(x) \left(\int_{Y_f} \frac{\partial(\alpha G_{ij})}{\partial y_j}(x, y) dy \right) dx = \int_{\Omega \times \Gamma} \alpha \gamma v (\gamma \varphi - (\gamma \nu \varphi) \nu) \quad (4.6)$$

and the proof is completed. \blacksquare

Let us introduce the so-called local solutions. Denoting

$$V_f = \{\varphi \in (H_{\text{per}}^1(Y_f)/\mathbb{R})^N, \operatorname{div}_y \varphi = 0\}, \quad (4.7)$$

$$K_f = \{\varphi \in (H_{\text{per}}^1(Y_f)/\mathbb{R})^N, \operatorname{div}_y \varphi = -1\} \quad (4.8)$$

we define $W^{kh} \in V_f$, $k, h \in \{1, 2, \dots, N\}$ and $W \in K_f$ as the unique solutions of the problems:

$$\int_{Y_f} (\delta_{ik} \delta_{jh} + e_{y,ij}(W^{kh})) e_{y,ij}(\psi) = 0, \quad \forall \psi \in V_f. \quad (4.9)$$

$$\int_{Y_f} e_y(W) e_y(\psi) = 0, \quad \forall \psi \in V_f. \quad (4.10)$$

The existence and uniqueness results for (4.9) are obtained by the Lax-Milgram Theorem. In order to prove (4.10) we notice that W is the projection of 0 on the closed convex $K_f \neq \emptyset$ in $(H_{\text{per}}^1(Y_f)/\mathbb{R})^N$ and the result follows.

Theorem 4.2. *The problem (4.3)–(4.4) is well-posed.*

Proof. Let $q \in M$; as $\tilde{q} \in L_0^2(\Omega)$, there exists $u_0 \in H_0^1(\Omega)^N$ with the properties

$$\operatorname{div} u_0 = \tilde{q} \quad (4.11)$$

$$\exists c_1 > 0 \quad \text{such that} \quad |u_0|_{H_0^1(\Omega)^N} \leq c_1 |\tilde{q}|_{L^2(\Omega)} \quad (4.12)$$

Similarly, as $q^f - \tilde{q}^f \in L^2(\Omega, L_0^2(Y_f))$, there exists $v_0 \in L^2(\Omega, H_0^1(Y_f)^N)$ with the properties

$$\operatorname{div}_y v_0 = q^f - \tilde{q}^f \quad (4.13)$$

$$\exists c_2 > 0 \quad \text{such that} \quad |v_0|_{L^2(\Omega, H_0^1(Y_f)^N)} \leq c_2 |q^f - \tilde{q}^f|_{L^2(\Omega \times Y_f)} \quad (4.14)$$

Denoting $w_0 = (v_0 + (\tilde{q} - \tilde{q}^f) W) \in L^2(\Omega, H_{\text{per}}^1(Y_f)^N)$, where W is the local solution defined by (4.10), we obtain

$$b(q, (u_0, w_0)) = |q|_{L^2(\Omega \times Y)}^2 \quad (4.15)$$

$$\begin{aligned}
|(u_0, w_0)|_X &\leq C \left(|u_0|_{H_0^1(\Omega)^N} + |v_0|_{L^2(\Omega, H_0^1(Y_f)^N)} + |\tilde{q} - \tilde{q}^f|_{L^2(\Omega)} \right) \leq \\
&\leq C |q|_{L^2(\Omega \times Y)}^2
\end{aligned} \tag{4.16}$$

and the inf-sup condition of b is obviously satisfied on $M \times X$.

We prove now that

$$X_0 = \{(u, w) \in X, \quad b(q, (u, w)) = 0, \quad \forall q \in M\}. \tag{4.17}$$

If $(u, w) \in X_0$, then for any $q \in M$,

$$\begin{aligned}
b(q, (u, w)) &= \int_{\Omega \times Y_s} q^s \operatorname{div}_x u - \int_{\Omega \times \Gamma} q^s (w \cdot \nu) = \\
&= \int_{\Omega} q^s (1 - m) \operatorname{div} \tilde{u}^s - \int_{\Omega} q^s \int_{Y_f} \operatorname{div}_y w = \int_{\Omega} q^s \operatorname{div} \tilde{u} = 0.
\end{aligned}$$

Conversely, if $(u, w) \in X$ with the property that $b(q, (u, w)) = 0, \forall q \in M$, then, choosing $q^s = 0$ it follows that there exists $C_1 \in \mathbb{R}$ such that

$$\operatorname{div} u^f + \operatorname{div} w = C_1 \quad \text{in} \quad \Omega \times Y_f$$

and consequently

$$\begin{aligned}
(1 - m) \operatorname{div} \tilde{u}^s - \int_{\Gamma} \gamma_{\nu} w &= C_1 (1 - m) \quad \text{in} \quad \Omega \\
(1 - m) \operatorname{div} \tilde{u}^s + m \operatorname{div} u^f &= C_1 \quad \text{in} \quad \Omega.
\end{aligned}$$

Integrating this last relation on Ω we finally get $C_1 = 0$.

As we have

$$\int_{\Omega \times \Gamma} \alpha (\gamma v - (\gamma_{\nu} v) \nu) \gamma v \geq \alpha_0 |\gamma v - (\gamma_{\nu} v) \nu|_{L^2(\Omega \times \Gamma)}^2 \geq 0$$

the X_0 -ellipticity of a in X is obvious and the proof is completed by Corollary 4.1, Ch.1 of [11]. \blacksquare

By a straightforward test we have:

Theorem 4.3. *If $((u, w), p) \in X$ is the solution of (4.3)–(4.4) then w is uniquely determined with respect to $u^f \in H_0^1(\Omega)$ by*

$$w(x, y) = W^{ij}(y) e_{ij}(u^f)(x) + W(y) \operatorname{div} u^f(x). \tag{4.18}$$

Finally, we eliminate w from (4.3)–(4.4) and find the corresponding classical homogenized problem verified by $(u, p^s) \in H \times \tilde{H}^1(\Omega)$:

To find $(u, p^s) \in V \times \tilde{H}^1(\Omega)$ such that

$$\int_{\Omega \times Y_s} Au\varphi + B \int_{\Omega} e(u^f) e(\varphi^f) + C \int_{\Omega} u^f \varphi^f = \int_{\Omega} (g - \nabla p^s) \tilde{\varphi}, \quad \forall \varphi \in H, \quad (4.19)$$

where the so-called effective coefficients which appear in (4.19) are given by

$$\lambda = \int_{Y_f} e_y(W) e_y(W) > 0 \quad (4.20)$$

$$\begin{aligned} B_{ijkh} &= \beta^2 \int_{Y_f} (\delta_{ki} \delta_{hj} + e_{y,ij}(W^{kh})) + \beta^2 \lambda \delta_{ik} \delta_{jh} = \\ &= \beta^2 \int_{Y_f} (\delta_{\ell k} \delta_{mh} + e_{y,\ell m}(W^{kh})) (\delta_{\ell i} \delta_{mj} + e_{y,\ell m}(W^{ij})) + \lambda \beta^2 \delta_{ik} \delta_{jh}. \end{aligned} \quad (4.21)$$

$$C_{ij} = \beta \int_{\Gamma} \alpha(y) (\delta_{ij} - \nu_i(y) \nu_j(y)) d\sigma_y \quad (4.22)$$

Remark 4.4. *The tensor B_{ijkh} is positive-definite and has the usual symmetry properties $B_{ijkh} = B_{khij} = B_{jikh}$. Also, we have to notice that*

$$C_{ij} \int_{\Omega} \varphi_i \varphi_j = \int_{\Omega \times \Gamma} \alpha(\gamma \varphi - (\gamma \nu \varphi) \nu)^2 \geq 0, \quad \forall \varphi \in H_0^1(\Omega)^N. \quad (4.23)$$

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